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ALMOST COMPLEX STRUCTURES ON TRIPLE BLOW-UPS

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Abstract. We show that for certain smooth 4-manifolds, almost complex structures on its triple blow-up induce all its spin^c structures with a non-negative-dimensional Seiberg–Witten moduli space. Since this includes all basic classes, our result provides an approximation to the simple type conjecture. Moreover, we expand a result by Dieter Kotschick showing that every finitely presented group is the fundamental group of a closed almost complex 4-manifold. We show that it can furthermore be chosen to have every even integer as the virtual dimension of the Seiberg–Witten moduli space of a spin^c structure.

Introduction

Seiberg–Witten theory is the application of the eponymous gauge theory to smooth 4-manifolds M, as described by Edward Witten in [Wit94]. It first requires a spin^c structure $\mathfrak{s}$ to formulate the Seiberg–Witten equations. It then considers the Seiberg–Witten moduli space of all solutions up to gauge transformation. Such a moduli space is associated a virtual dimension $d(\mathfrak{s}) \in \mathbb{Z}$, dictated purely by the topology of M and the spin^c structure $\mathfrak{s}$. Almost complex structures are naturally related to this construction: every almost complex structure J induces a spin^c structure $\mathfrak{s}_J$ and furthermore $d(\mathfrak{s}_J) = 0$ (see [Nic00, p. 459; Sco05, p. 390]). In reverse, every such spin^c structure is induced by an almost complex structure, giving a surjection.

Our main result provides generalized surjections between spin^c structures with inequality conditions for the virtual dimension. It is obtained by combining connected sums with $\overline{\mathbb{C}P^2}$ and $\mathbb{C}P^2$ with Gauss' Eureka theorem (see [Ang95, Thrm. 6.5.5]) from number theory (assuming $0 \in \mathbb{N}$). In particular, it extends the previous statement by showing that every spin^c structure $\mathfrak{s}$ with $d(\mathfrak{s}) \geq 0$ is induced by an almost complex structure on the triple blow-up, meaning a connected sum of the 4-manifold with $3\overline{\mathbb{C}P^2}$. Its main significance comes from its uniformity. Namely, no matter the manifold or the virtual dimension, three blow-ups are always sufficient to guarantee this and are optimal in general.

A direct consequence of this result concerns the (Seiberg–Witten) simple type conjecture. It claims that for every oriented closed smooth 4-manifold with $-b_1 + b_2^+$ odd and $b_2^+ \geq 2$, every (Seiberg–Witten) basic class $\mathfrak{s}$ (a spin^c structure with a non-trivial Seiberg–Witten invariant) has $d(\mathfrak{s}) = 0$ or equivalently is induced by an almost complex structure [Wit94, p. 27] (see also [Nic00, p. 150; Sco05, p. 402]). Since a basic class $\mathfrak{s}$ requires $d(\mathfrak{s}) \geq 0$ by definition, our result can be applied.

Moreover, we also consider the case when the orientation of two blow-ups is changed, meaning a connected sum of the 4-manifold with $2\mathbb{C}P^2\#\overline{\mathbb{C}P^2}$. We show

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that this removes all inequality conditions for the virtual dimension: every spin^c structure and every even number can be combined to a spin^c structure on this connected sum. In partiular, every spin^c structure comes from an almost complex structure on this connected sum. We use this to extend a result by Dieter Kotschick in [Kot92], stating that every finitely presented group is the fundamental group of a closed almost complex 4-manifold. His result was obtained analogous by combining connected sums with $\overline{\mathbb{C}P^2}$ and $\mathbb{C}P^2$ with Lagrange's four-squares theorem (see [Ang95, Thrm. 1.8.4]) from number theory. Both Gauss' and Lagrange's theorem are special cases of Fermat's polygonal number theorem (see [Ang95, Thrm. 6.6.3]), giving further common ground.

The paper is organized as follows: Section 1 will state the main result concerning the generalized surjections and their implications. Section 2 will review their setting and Section 3 will provide their proofs. Section 4 will discuss aspects like potential improvement, relation to other works and relevant details concerning the simple type conjecture.

1. Results

Throughout this paper, all manifolds are assumed to be connected. Let $M\#N$ be the connected sum of smooth 4-manifolds M and N. It is obtained by removing an embedded 4-disc from both, glueing the boundary spheres together and uniquely smoothing it (see [Sco05, p. 12]). Collapsing one of the manifolds to a point and smoothing the map by homotopy yields canonical projections $M\#N \twoheadrightarrow M, N$.

Let M be an oriented closed smooth 4-manifold with $(-b_1 + b_2^+)(M)$ odd. Let $\mathrm{Spin}^c(M)$ denote the set of spin^c structures of M. The main result of the paper is the following:

Theorem 1. *For every $n \in \mathbb{N}$, the canonical projections $p\colon M\#(n+3)\overline{\mathbb{C}P^2} \twoheadrightarrow M$ and $q\colon M\#(2n+4)\mathbb{C}P^2 \twoheadrightarrow M$ induce surjections:*

$$(p^*, d)\colon \mathrm{Spin}^c\left(M\#(n+3)\overline{\mathbb{C}P^2}\right) \twoheadrightarrow \{(\mathfrak{s}, 2m) \in \mathrm{Spin}^c(M) \times 2\mathbb{Z} | d(\mathfrak{s}) \geq 2m\}\,; \quad (1)$$

$$\begin{aligned}(q^*, d)\colon \mathrm{Spin}^c&\left(M\#(2n+4)\mathbb{C}P^2\right)\\ &\twoheadrightarrow \{(\mathfrak{s}, 2m) \in \mathrm{Spin}^c(M) \times 2\mathbb{Z} | d(\mathfrak{s}) \leq 2(m+n+2)\}\,.\end{aligned} \quad (2)$$

Induced means that the spin^c structure on the connected sum is restricted onto one of the manifolds with an embedded 4-disc removed and then uniquely extended, using that the 3-sphere has a unique spin^c structure.

Let $\mathcal{J}(M)$ denote the set of almost complex structures of M. For every $m \in \mathbb{Z}$, let $\mathrm{Spin}^c_m(M)$, $\mathrm{Spin}^c_{\geq m}(M)$ and $\mathrm{Spin}^c_{\leq m}(M)$ be the subsets of $\mathrm{Spin}^c(M)$ composed of spin^c structures $\mathfrak{s}$ with $d(\mathfrak{s}) = m$, $d(\mathfrak{s}) \geq m$ and $d(\mathfrak{s}) \leq m$, respectively.

As shown by Wu in [Wu52] and Hirzebruch–Hopf in [HH58] (before the advent of Seiberg–Witten theory), all spin^c structures with a zero-dimensional Seiberg–Witten moduli space are induced by almost complex structures, which gives a surjection (see [Sco05, p. 421]):

$$\mathrm{ind}\colon \mathcal{J}(M) \twoheadrightarrow \mathrm{Spin}^c_0(M), J \mapsto \mathfrak{s}_J. \quad (3)$$

For $m = 0$, Theorem 1 combined with this result gives analogous surjections:

$$p^* \circ \mathrm{ind}\colon \mathcal{J}\left(M\#(n+3)\overline{\mathbb{C}P^2}\right) \twoheadrightarrow \mathrm{Spin}^c_{\geq 0}(M);$$

$$q^* \circ \mathrm{ind} \colon \mathcal{J}\left(M\#(2n+4)\mathbb{C}P^2\right) \twoheadrightarrow \mathrm{Spin}^{\mathrm{c}}_{\leq 2n+4}(M).$$

In the former, increasing the number of blow-ups has no effect on the target, which concerns exactly the spin$^{\mathrm{c}}$ structures relevant in Seiberg–Witten theory. For $n = 0$, Theorem 1 further reduces to its strongest special case:

Corollary 2. *The canonical projection $p \colon M\#3\overline{\mathbb{C}P^2} \twoheadrightarrow M$ induces a surjection:*

$$p^* \circ \mathrm{ind} \colon \mathcal{J}\left(M\#3\overline{\mathbb{C}P^2}\right) \twoheadrightarrow \mathrm{Spin}^{\mathrm{c}}_{\geq 0}(M). \tag{4}$$

First, its implication for the simple type conjecture is:

Theorem 3. *Every basic class of an oriented closed smooth 4-manifold with $-b_1+b_2^+$ odd is induced by an almost complex structure on the triple blow-up.*

Second, its adaption with the orientation of two blow-ups changed is:

Lemma 4. *The canonical projection $p \colon M\#2\mathbb{C}P^2\#\overline{\mathbb{C}P^2} \twoheadrightarrow M$ induces surjections:*

$$(p^*, d) \colon \mathrm{Spin}^{\mathrm{c}}\left(M\#2\mathbb{C}P^2\#\overline{\mathbb{C}P^2}\right) \twoheadrightarrow \mathrm{Spin}^{\mathrm{c}}(M) \times 2\mathbb{Z}; \tag{5}$$

$$p^* \circ \mathrm{ind} \colon \mathcal{J}\left(M\#2\mathbb{C}P^2\#\overline{\mathbb{C}P^2}\right) \twoheadrightarrow \mathrm{Spin}^{\mathrm{c}}(M). \tag{6}$$

Combining Lemma 4 with Kotschick's result yields:

Theorem 5. *For every finitely presented group G there exists a closed almost complex 4-manifold M with $\pi_1(M) \cong G$ so that $d \colon \mathrm{Spin}^{\mathrm{c}}(M) \twoheadrightarrow 2\mathbb{Z}$ is surjective.*

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2. Setting

For an oriented closed smooth 4-manifold M, its intersection form canonically pairs the cup product of two second cohomology classes with the fundamental class $[M] \in H_4(M, \mathbb{Z}) \cong \mathbb{Z}$ (see [FU91, p. 20; DK97, p. 2]):

$$Q_M \colon H^2(M, \mathbb{Z}) \times H^2(M, \mathbb{Z}) \to \mathbb{Z}, (x, y) \mapsto xy[M] := \langle x \smile y, [M] \rangle.$$

Let $b_k(M) := \operatorname{rank} H^k(M, \mathbb{Z}) \in \mathbb{N}$ be the Betti numbers with Poincaré duality assuring $b_k(M) = b_{4-k}(M)$. Since the intersection form vanishes for all torsion elements, it reduces to a bilinear form on the free part $H^2(M, \mathbb{Z})/\mathrm{tor} \cong \mathbb{Z}^{b_2(M)}$. It can therefore be expressed as a symmetric matrix $Q_M \in \mathbb{Z}^{b_2(M) \times b_2(M)}$ with $\det(Q_M) = \pm 1$. Let $b_2^{\pm}(M)$ denote the rank of maximally positive and maximally negative subspaces, respectively. One then has a splitting $b_2(M) = (b_2^+ + b_2^-)(M)$. Two important topological invariants are the signature and Euler characteristic (see [FU91, p. 22; DK97, p. 3]), given respectively by:

$$\sigma(M) := (b_2^+ - b_2^-)(M) \in \mathbb{Z}; \tag{7}$$

$$\chi(M) := 2 - 2b_1(M) + b_2(M) \in \mathbb{Z}. \tag{8}$$

One has $Q_{\mathbb{C}P^2} = [+1]$ and $Q_{\overline{\mathbb{C}P^2}} = [-1]$ (see [FU91, p. 21; DK97, p. 3]). Therefore $\sigma(\mathbb{C}P^2) = +1$ and $\sigma(\overline{\mathbb{C}P^2}) = -1$. One also has $\chi(\mathbb{C}P^2) = \chi(\overline{\mathbb{C}P^2}) = 3$.

An almost complex structure $J \in \mathcal{J}(M)$ is composed of a complex plane bundle $E \twoheadrightarrow M$ and a real vector bundle isomorphism $E|_{\mathbb{R}} \cong TM$. Its first Chern class

$c_1(J) := c_1(E) \in H^2(M, \mathbb{Z})$ fulfills $c_1(J) \bmod 2 = w_2(M)$ and (see [GS99, Thrm. 1.4.15; Sco05, p. 377]):

$$c_1^2(J)[M] = (3\sigma + 2\chi)(M). \tag{9}$$

Any such class is induced by an almost complex structure.

A spin^c structure $\mathfrak{s} \in \mathrm{Spin}^c(M)$ is composed of two complex plane bundles $W^{\pm} \twoheadrightarrow M$ with $\det(W^+) = \det(W^-)$ and a complex vector bundle isomorphism $TM \otimes \mathbb{C} \cong \underline{\mathrm{Hom}}(W^+, W^-)$ satisfying the Clifford condition. Its first Chern class $c_1(\mathfrak{s}) := c_1(W^{\pm}) \in H^2(M, \mathbb{Z})$ fulfills $c_1(\mathfrak{s}) \bmod 2 = w_2(M)$. Any such class is induced by a spin^c structure, which is always unique if $H^2(M, \mathbb{Z})$ has no 2-torsion.

Every almost complex structure J induces a spin^c structure $\mathfrak{s}_J$, which establishes the map (3). One has $c_1(J) = c_1(\mathfrak{s}_J)$, which serves as a unique description of (3) if $H^2(M, \mathbb{Z})$ has no 2-torsion. However, not every spin^c structure $\mathfrak{s}$ is induced by an almost complex structure. Its obstruction comes from (9) and is also reflected in the virtual dimension of its Seiberg–Witten moduli space [Wit94, Eq. (2.5)] (see also [GS99, Thrm. 2.4.24; Sco05, p. 397]):

$$d(\mathfrak{s}) = \frac{1}{4}\left(c_1^2(\mathfrak{s})[M] - (3\sigma + 2\chi)(M)\right) \in 2\mathbb{Z}. \tag{10}$$

It is even as van der Blij's lemma (see [Sco05, p. 170]) assures $c_1^2(\mathfrak{s})[M] - \sigma(M) \in 8\mathbb{Z}$ and (7) and (8) assure $(\sigma + \chi)(M) = 2 + 2(-b_1 + b_2^+)(M) \in 2\mathbb{Z}$, respectively.

Remark 6. *One has (see [GS99, Ex. 1.3.5; Sco05, p. 118, 120]):*

$$Q_{M\#n\overline{\mathbb{C}P^2}} \cong Q_M \oplus n[-1]; \tag{11}$$

$$Q_{M\#n\mathbb{C}P^2} \cong Q_M \oplus n[+1]; \tag{12}$$

$$(3\sigma + 2\chi)\left(M\#n\overline{\mathbb{C}P^2}\right) = (3\sigma + 2\chi)(M) - n; \tag{13}$$

$$(3\sigma + 2\chi)\left(M\#n\mathbb{C}P^2\right) = (3\sigma + 2\chi)(M) + 5n. \tag{14}$$

Let $x \in H^2(\mathbb{C}P^2, \mathbb{Z}) \cong H^2(\overline{\mathbb{C}P^2}, \mathbb{Z}) \cong \mathbb{Z}$ be a generator, meaning $x^2[\mathbb{C}P^2] = 1$ and $x^2[\overline{\mathbb{C}P^2}] = -1$. For $a \in \mathbb{Z}$, let $\bar{\mathfrak{s}}_a \in \mathrm{Spin}^c(\overline{\mathbb{C}P^2})$ be the spin^c structure with $c_1(\bar{\mathfrak{s}}_a) = (2a+1)x$ and let $\mathfrak{s}_a \in \mathrm{Spin}^c(\mathbb{C}P^2)$ be the spin^c structure with $c_1(\mathfrak{s}_a) = (2a+1)x$. One has:

$$c_1^2(\bar{\mathfrak{s}}_a)\left[\overline{\mathbb{C}P^2}\right] = -(2a+1)^2 = -4a(a+1) - 1; \tag{15}$$

$$c_1^2(\mathfrak{s}_a)[\mathbb{C}P^2] = (2a+1)^2 = 4a(a+1) + 1. \tag{16}$$

Let $\bar{\mathfrak{s}}_{-1,0} \in \{\bar{\mathfrak{s}}_{-1}, \bar{\mathfrak{s}}_0\}$, $\mathfrak{s}_{-1,0} \in \{\mathfrak{s}_{-1}, \mathfrak{s}_0\}$ and $\mathfrak{s}_{-2,1} \in \{\mathfrak{s}_{-2}, \mathfrak{s}_1\}$, then one has $c_1^2(\bar{\mathfrak{s}}_{-1,0})[\overline{\mathbb{C}P^2}] = -1$, $c_1^2(\mathfrak{s}_{-1,0})[\mathbb{C}P^2] = 1$ and $c_1^2(\mathfrak{s}_{-2,1})[\mathbb{C}P^2] = 9$, respectively.

Lemma 7. *For an almost complex 4-manifold M:*

a) *$M\#\overline{\mathbb{C}P^2}$ is almost complex.*
b) *$M\#2\mathbb{C}P^2$ is almost complex.*

Proof. Let $J \in \mathcal{J}(M)$. Using (9), (11), (13) and (15):

$$(c_1(J)\#\bar{\mathfrak{s}}_{-1,0})^2\left[M\#\overline{\mathbb{C}P^2}\right] = (3\sigma + 2\chi)(M\#\overline{\mathbb{C}P^2}). \tag{17}$$

Analogous from (9), (12), (14) and (16):

$$(c_1(J)\#c_1(\mathfrak{s}_{-1,0})\#c_1(\mathfrak{s}_{-2,1}))^2[M\#2\mathbb{C}P^2] = (3\sigma + 2\chi)(M\#2\mathbb{C}P^2). \tag{18}$$

Hence $c_1(J)\#c_1(\bar{\mathfrak{s}}_{-1,0})$ and $c_1(J)\#c_1(\mathfrak{s}_{-1,0})\#c_1(\mathfrak{s}_{-2,1})$ fulfill (9). According to (3), both come from an almost complex structure. □

3. Proofs

Let M be an oriented smooth 4-manifold and $\mathfrak{s} \in \mathrm{Spin}^{\mathrm{c}}(M)$. Using (10), (11), (13) and (15) (see [Nic00, p. 459]):

$$d(\mathfrak{s}\#\overline{\mathfrak{s}}_a) = d(\mathfrak{s}) - a(a+1). \tag{19}$$

Analogous from (10), (12), (14) and (16):

$$d(\mathfrak{s}\#\mathfrak{s}_a) = d(\mathfrak{s}) + a(a+1) - 1. \tag{20}$$

Relying on the virtual dimension of all Seiberg–Witten moduli spaces being even, the core of the proofs is the combination of (19) and (20) with the following result from number theory:

Theorem 8 (Gauss' Eureka theorem)**.** *For every $n \in \mathbb{N}$, there exist $a, b, c \in \mathbb{N}$ with:*

$$2n = a(a+1) + b(b+1) + c(c+1). \tag{21}$$

Proof of Theorem 1. Let $\mathfrak{s} \in \mathrm{Spin}^{\mathrm{c}}(M)$ and $m \in \mathbb{Z}$ fulfill $2m \leq d(\mathfrak{s})$. According to Theorem 8, there exist $a, b, c \in \mathbb{N}$ such that:

$$d(\mathfrak{s}) - 2m = a(a+1) + b(b+1) + c(c+1). \tag{22}$$

For $\widehat{\mathfrak{s}} = \mathfrak{s}\#\overline{\mathfrak{s}}_a\#\overline{\mathfrak{s}}_b\#\overline{\mathfrak{s}}_c \in \mathrm{Spin}^{\mathrm{c}}(M\#3\overline{\mathbb{C}P^2})$, applying (19) to each of the three added spin$^{\mathrm{c}}$ structures makes it a preimage:

$$(p^*, d)(\widehat{\mathfrak{s}}) = (\mathfrak{s}, 2m).$$

Hence the map (1) is surjective. It is also well-defined and furthermore both for $n \geq 0$ since (19) with $a(a+1) \geq 1$ and the minimum attained for $a = -1, 0$ yields:

$$d(\widehat{\mathfrak{s}}\#n\overline{\mathfrak{s}}_{-1,0}) = d(\widehat{\mathfrak{s}}) \leq d(\mathfrak{s}).$$

Let $m \in \mathbb{Z}$ fulfill $d(\mathfrak{s}) \leq 2m + 4$. According to Theorem 8, there exist $a, b, c \in \mathbb{N}$ such that:

$$2m + 4 - d(\mathfrak{s}) = a(a+1) + b(b+1) + c(c+1). \tag{23}$$

For $\widehat{\mathfrak{s}} = \mathfrak{s}\#\mathfrak{s}_a\#\mathfrak{s}_b\#\mathfrak{s}_c\#\mathfrak{s}_{-1,0} \in \mathrm{Spin}^{\mathrm{c}}(M\#4\mathbb{C}P^2)$, applying (20) to each of the four added spin$^{\mathrm{c}}$ structures makes it a preimage:

$$(q^*, d)(\widehat{\mathfrak{s}}) = (\mathfrak{s}, 2m).$$

Hence the map (2) is surjective. It is also well-defined and furthermore both for $n \geq 0$ since (20) with $a(a+1) \geq 1$ and the minimum attained for $a = -1, 0$ yield:

$$d(\widehat{\mathfrak{s}}\#2n\mathfrak{s}_{-1,0}) + 2n \geq d(\widehat{\mathfrak{s}}) \geq d(\mathfrak{s}) - 4.$$

□

Proof of Lemma 4. If the sign of the entire last summand in (21) is changed, then it directly holds for every $n \in \mathbb{Z}$ using $a = -1, 0$, $b = n$ and $c = n - 1$. Let $\mathfrak{s} \in \mathrm{Spin}^{\mathrm{c}}(M)$, $m \in \mathbb{Z}$ and:

$$n = -\frac{d(\mathfrak{s})}{2} + m + 1.$$

For $\widehat{\mathfrak{s}} = \mathfrak{s}\#\mathfrak{s}_{-1,0}\#\mathfrak{s}_n\#\overline{\mathfrak{s}}_{n-1} \in \mathrm{Spin}^{\mathrm{c}}\left(M\#2\mathbb{C}P^2\#\overline{\mathbb{C}P^2}\right)$, applying (19) once and (20) twice makes it a preimage:

$$(p^*, d)(\widehat{\mathfrak{s}}) = (\mathfrak{s}, 2m).$$

□

Proof of Theorem 5. Let G be a finitely presented group. According to [Kot92], there exists an almost complex closed 4-manifold M with $\pi_1(M) \cong G$. According to Lemma 7 (or implied by Lemma 4), $M\#2\mathbb{C}P^2\#\overline{\mathbb{C}P^2}$ is also an almost complex closed 4-manifold. Furthermore $\pi_1(M\#2\mathbb{C}P^2\#\overline{\mathbb{C}P^2}) \cong G$ since $\mathbb{C}P^2$ and $\overline{\mathbb{C}P^2}$ are simply connected. Lemma 4 then yields the claim. □

4. Discussion

John A. Ewell provided two results related to Theorem 8: In [Ewe92a], he gives a criterium for special cases reducing it to two summands, which transfers directly to reducing Corollary 2 to two blow-ups. However, this requires considering prime factors, which for arbitrary differences like in (22) and (23) are completely unknown. In [Ewe92b], he gives an explicit formula for how many possible decompositions there are, which transfers directly to how much Corollary 2 deviates from injectivity.

Remark 9. *Three blow-ups in Corollary 2 is optimal in general. For instance, for $n = 5$, one has $a = 1$, $b = 1$ and $c = 2$ in (21), making it the smallest example of Theorem 8 for which no $a, b, c \in \mathbb{N}$ can vanish. For $M = \mathbb{C}P^2$, using (10) and (16), one has $d(\mathfrak{s}_{-4,3}) = 10$ (see also [Nic00, p. 168]). From (10), (12) and (13) one has:*

$$\begin{aligned} d(\mathfrak{s}_{-4,3}\#\mathfrak{s}_a\#\mathfrak{s}_b) &= \frac{1}{4}\left(c_1^2(\mathfrak{s}_{-4,3}\#\mathfrak{s}_a\#\mathfrak{s}_b)\left[\mathbb{C}P^2\#2\overline{\mathbb{C}P^2}\right] - (3\sigma + 2\chi)\left(\mathbb{C}P^2\#2\overline{\mathbb{C}P^2}\right)\right) \\ &= 10 - a(a+1) - b(b+1) \neq 0. \end{aligned}$$

Hence no $a, b \in \mathbb{N}$ exist so that $\mathfrak{s}_{-4,3}\#\mathfrak{s}_a\#\mathfrak{s}_b \in \mathrm{Spin}^c(\mathbb{C}P^2\#2\overline{\mathbb{C}P^2})$ fulfills (9) and is therefore induced by an almost complex structure according to (3).

It is also worth noting that both results have been used in other fields using the same concepts as this paper: For instance, they were used to study circle actions on compact almost complex manifolds by Godinho–Pelayo–Sabatini in [GPS14], almost complex closed manifolds with the same rational cohomology as a connected sum of $\mathbb{H}P^2$ and $\overline{\mathbb{H}P^2}$ by Milivojević in [Mil22], generalizations of spin^c structures by Albanese–Milivojević in [AM22], and string^c structures by Duan–Han–Huang in [DHH20], among others.

Two further remarks are important concerning Theorem 3. First, the simple type conjecture requires $b_2^+ \geq 2$, which cannot be omitted. Otherwise, $\mathbb{C}P^2$ provides a counterexample since it fulfills all other conditions, but has $b_2^+(\mathbb{C}P^2) = 1$ and basic classes $\mathfrak{s}$ with $d(\mathfrak{s}) > 0$ (see [Nic00, p. 168]). However, Theorem 3 does omit $b_2^+ \geq 2$. Second, blow-ups with the spin^c structures $\mathfrak{s}_{-1,0}$ preserve the Seiberg–Witten invariant while all others cause it to vanish (see [GS99, Thrm. 2.4.9]). However, in general these are considered in the proofs. Both of these aspects are based on the fact that Seiberg–Witten invariants, which require $b_2^+ \geq 2$ for metric independence and which are at the core of the simple type conjecture, were not used to derive Theorem 3. An immediate question is then whether Theorem 3 can be improved by using the Seiberg–Witten invariants and Ewell's results to obtain an even closer approximation to the simple type conjecture:

Conjecture 10. *Every basic class of an oriented closed smooth 4-manifold with $-b_1 + b_2^+$ odd and $b_2^+ \geq 2$ is induced by an almost complex structure on the double blow-up.*

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