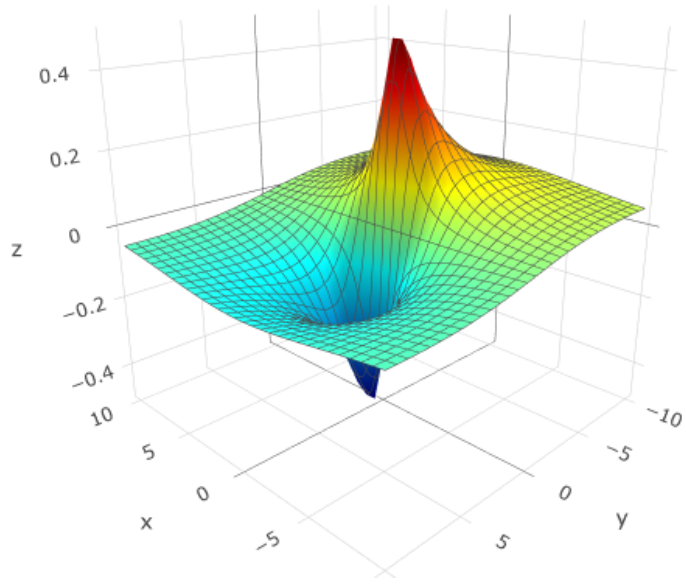
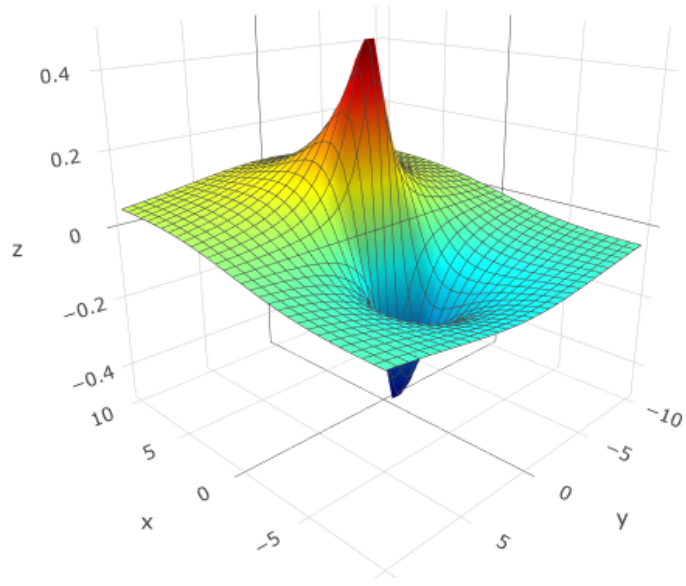
Yang–Mills equations

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0.4
0.2
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z
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0.4
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- General relativity is SO(1,3) Yang–Mills theory

Basics of Yang–Mills theory

Let G be a Lie group with Lie algebra $\mathfrak{g}$. Yang–Mills theory is formulated for a smooth principal G-bundle $P \twoheadrightarrow M$ over a Riemannian manifold, in particular for *principal connections* whose space is denoted:

$$\Omega^1_{\mathrm{conn}}(P) := \Omega^1(M, \mathrm{Ad}(P)) \cong \Omega^1_{\mathrm{Ad}}(P, \mathfrak{g}). \tag{1}$$

If the base manifold is not globally the euclidean space, but only locally, then consider the restrictions to suitable open subsets.

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If the principal bundle is not globally trivial, but only locally, then consider the restriction to suitable open subsets.

su(2)-valued 0-form

Let $\Phi \in \Omega^0(\mathbb{R}^3, \mathfrak{su}(2))$, then one can write:

$$\Phi = \Phi^1 \otimes \sigma_1 + \Phi^2 \otimes \sigma_2 + \Phi_3 \otimes \sigma_3. \tag{2}$$

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The wedge product with itself vanishes:

$$\begin{aligned}
[\Phi, \Phi] = \Phi^2\Phi^3 \otimes \underbrace{[\sigma_2, \sigma_3]}_{=\sigma_1} + \Phi^3\Phi^2 \otimes \underbrace{[\sigma_3, \sigma_2]}_{=-\sigma_1} \\
+ \Phi^3\Phi^1 \otimes \underbrace{[\sigma_3, \sigma_1]}_{=\sigma_2} + \Phi^1\Phi^3 \otimes \underbrace{[\sigma_1, \sigma_3]}_{=-\sigma_2} \\
+ \Phi^1\Phi^2 \otimes \underbrace{[\sigma_1, \sigma_2]}_{=\sigma_3} + \Phi^2\Phi^1 \otimes \underbrace{[\sigma_2, \sigma_1]}_{=-\sigma_3} = 0.
\end{aligned} \tag{3}$$

su(2)-valued 1-form

Let $A \in \Omega^1(\mathbb{R}^3, \mathfrak{su}(2))$, then one can write:

$$\begin{aligned} A = &(A_1^1 \mathrm{d}x + A_2^1 \mathrm{d}y + A_3^1 \mathrm{d}z) \otimes \sigma_1 \\ &+ (A_1^2 \mathrm{d}x + A_2^2 \mathrm{d}y + A_3^2 \mathrm{d}z) \otimes \sigma_2 \\ &+ (A_1^3 \mathrm{d}x + A_2^3 \mathrm{d}y + A_3^3 \mathrm{d}z) \otimes \sigma_3. \end{aligned} \tag{4}$$

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The wedge product with doesn't vanish:

$$\begin{aligned} \frac{1}{2}[A \wedge A] = &\big((A_2^2 A_3^3 - A_2^3 A_3^2)\mathrm{d}y \wedge \mathrm{d}z - (A_1^2 A_3^3 - A_1^3 A_3^2)\mathrm{d}z \wedge \mathrm{d}x \\ &+ (A_1^2 A_2^3 - A_1^3 A_2^2)\mathrm{d}x \wedge \mathrm{d}y\big) \otimes \sigma_1 \\ &+ \big(- (A_2^1 A_3^3 - A_2^3 A_3^1)\mathrm{d}y \wedge \mathrm{d}z + (A_1^1 A_3^3 - A_1^3 A_3^1)\mathrm{d}z \wedge \mathrm{d}x \\ &- (A_1^1 A_2^3 - A_1^3 A_2^1)\mathrm{d}x \wedge \mathrm{d}y\big) \otimes \sigma_2 \\ &+ \big((A_2^1 A_3^2 - A_2^2 A_3^1)\mathrm{d}y \wedge \mathrm{d}z - (A_1^1 A_3^2 - A_1^2 A_3^1)\mathrm{d}z \wedge \mathrm{d}x \\ &+ (A_1^1 A_2^2 - A_1^2 A_3^1)\mathrm{d}x \wedge y\big) \otimes \sigma_3. \end{aligned} \tag{5}$$

su(2)-valued 1-form

Considering the differential form A being represented by a matrix with minor matrices A', one can consider the Hodge dual of the wedge product with itself to be represented by the adjugate matrix:

$$\begin{aligned}\frac{1}{2} \star [A \wedge A] = &(\det(A'_{11})\mathrm{d}x - \det(A'_{12})\mathrm{d}y + \det(A'_{13})\mathrm{d}z) \otimes \sigma_1 \\ &+ (-\det(A'_{21})\mathrm{d}x + \det(A'_{22})\mathrm{d}y - \det(A'_{23})\mathrm{d}z) \otimes \sigma_2 \\ &+ (\det(A'_{31})\mathrm{d}x - \det(A'_{32})\mathrm{d}y + \det(A'_{33})\mathrm{d}z) \otimes \sigma_3. \quad (6)\end{aligned}$$

Curvature form

For a principal connection $A \in \Omega^1_{\mathrm{conn}}(P)$, let:

$$F_A := \mathrm{d}A + \frac{1}{2}[A \wedge A] \tag{7}$$

The first term is differential and describes the field strength. The second term is algebraic and describes the self-interaction.

Yang–Mills equations

A *flat connection* has vanishing curvature:

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One furthermore has the *Bianchi identity*:

$$\mathrm{d}_A F_A = 0. \tag{10}$$

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The four-dimensional Yang–Mills equations have two important BPS equations which are the self-dual and anti self-dual Yang–Mills ((A)SDYM) equations:

$$\star F_A = \pm F_A. \tag{11}$$

Chern–Weil theory

(Chern–)Gauß–Bonnet theorem: The Euler characteristic can be computed from the Gaussian curvature:

$$\chi(M) = \int_M K \tag{12}$$

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As a consequence, flat connections don't always exist. Yang–Mills connections are a suitable generalization, which always exists.

Yang–Mills action functional

For a principal connection $A \in \Omega^1_{\mathrm{conn}}(P)$, the *Yang–Mills action functional* is given by:

$$\mathrm{YM}(A) = \int_M \|F_A\|^2 \mathrm{d}\,\mathrm{vol}_g = \int_M \mathrm{tr}(F_A \wedge \star F_A)\mathrm{d}\,\mathrm{vol}_g \geq 0 \qquad (14)$$

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Flat connections are the global minima and Yang–Mills connections are the critical points, hence the local extrema.

Yang–Mills action functional

For a(n) (anti) self-dual connection $A \in \Omega^1_{\mathrm{conn}}(P)$, one has:

$$\begin{aligned} \mathrm{YM}(A) &= \int_M \mathrm{tr}(F_A \wedge \star F_A)\mathrm{d}\,\mathrm{vol}_g \\ &= \pm \int_M \mathrm{tr}(F_A \wedge F_A)\mathrm{d}\,\mathrm{vol}_g = \pm c_2(P)[M] \end{aligned} \tag{15}$$

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For any principal connection $A \in \Omega^1_{\mathrm{conn}}(P)$, one has:

$$\begin{aligned} \mathrm{YM}(A) &= \frac{1}{2}\int_M \|F_A\|^2 + \|\star F_A\|^2 \mathrm{d}\,\mathrm{vol}_g \\ &= \int_M \frac{1}{2}\|F_A \pm \star F_A\|^2 \mp \mathrm{tr}(F_A \wedge F_A)\mathrm{d}\,\mathrm{vol}_g \geq |c_2(P)[M]| \end{aligned} \tag{16}$$