Seiberg–Witten equations

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Dirac operator

A *Dirac operator* $\not{\partial}$ is a differential operator, which yields the Laplace operator Δ (or in physics the d'Alembert/wave operator $\Box = \partial_t^2 - \Delta$) when applied twice:

$$\Delta = \not{\partial}^2. \tag{1}$$

For example the Laplace operator on $C^2(\mathbb{C}^2, \mathbb{C}^2)$ has a Dirac operator:

$$\not{\partial}^2 = \begin{pmatrix} 0 & \partial_x + i\partial_y \\ \partial_x - i\partial_y & 0 \end{pmatrix}^2 = \mathbf{1}_2 \Delta. \tag{2}$$

Considering vector-valued coefficients, it is also possible to make the Nabla operator into a Dirac operator:

$$\nabla^2 = \begin{pmatrix} \begin{pmatrix} \partial_x \\ \partial_y \end{pmatrix} & 0 \\ 0 & \begin{pmatrix} \partial_x \\ \partial_y \end{pmatrix} \end{pmatrix}^2 = \mathbf{1}_2 \Delta \tag{3}$$

Dirac equation

A famous application of Dirac operators is in the *Dirac equation*, using a four-dimensional base manifold and four-dimensional *Dirac matrices* with:

$$(i\not{\partial} - m)(i\not{\partial} + m) = -(\Box + m^2). \tag{4}$$

Hence the Klein–Gordon equation $(\Box + m^2)\phi = 0$ for a scalar-valued field ϕ is implied for all components by the Dirac equation $(i\not{\partial} - m)\psi = 0$ for a vector-valued field ψ, called *Dirac spinor*. (Hence the Dirac equation is a Bogomolny–Prasad–Sommerfeld (BPS) equation for the Klein–Gordon equation.)

Dirac equation

An electromagnetic field can be considered by adding its vector potential A to the momentum, which under quantization becomes the partial derivative ∂ or ∇:

$$\not{\partial}_A := \not{\partial} - \not{A} \tag{5}$$

$$\nabla_A := \nabla - A \tag{6}$$

The square of the second operator is known from the *Schrödinger equation* in an electromagnetic field:

$$\nabla_A^2 = \Box - \nabla A - 2A\nabla - A^2. \tag{7}$$

In electrodynamics, one usually takes the Lorentz-invariant Lorenz gauge[1] $\nabla A = 0$ and in quantum mechanics, one usually uses perturbation theory in first order with $A^2 = 0$. The remaining term $A\nabla$ then yields the *Zeeman effect*, the splitting of spectral lines in a magnetic field according to their magnetic quantum number.

[1] Hendrik Antoon Lorentz and Ludvig Lorenz were different people.

Lichnerowicz formula

The square of the first operator can be computed with the *Lichnerowicz formula*, which in full generality is given by:

$$\not{\partial}_A^* \not{\partial}_A \psi = \nabla_A^* \nabla_A \psi + \frac{1}{4} R\psi + \frac{1}{2} F_A^+(\psi). \tag{8}$$

A star indicates the adjoint operator with repsect to the Riemannian metric of the base manifold. R is the *scalar curvature* (as it is usually called in mathematics) or *Ricci curvature* (as it is usually called in physics).

For a simplification, consider the scalar product $\langle x, y \rangle = x^T y$ on n-vectors, which with a $n \times n$ matrix A fulfills:

$$\langle A^T x, y \rangle = (A^T x) y = x^T A y = \langle x, Ay \rangle, \tag{9}$$

making A^T the adjoint operator of A.

(Anti) self-dual part

A curvature form F_A can be separated into a *self-dual* and *anti self-dual part*:

$$F_A^{\pm} := \frac{1}{2}\left(F_A \pm \star F_A\right) \tag{10}$$

with $\star F_A^{\pm} = \pm F_A$.

The self-dual Yang–Mills equations $\star F_A = +F_A$ can therefore be reformulated as $F_A^- = 0$ and the anti self-dual Yang–Mills equations $\star F_A = -F_A$ can therefore be reformulated as $F_A^+ = 0$.

For a simplification, consider any quadratic matrix M, which can be separated into a *symmetric* and *antisymmetric part*:

$$M_{\text{sym}} := \frac{1}{2}(M + M^T); \tag{11}$$

$$M_{\text{asym}} := \frac{1}{2}(M - M^T) \tag{12}$$

with $M_{\text{sym}}^T = M_{\text{sym}}$ and $M_{\text{asym}}^T = -M_{\text{asym}}$.

Seiberg–Witten equations

The *Seiberg–Witten equations* are given by the massless Dirac equation and an additional curvature equation:

$$\not{\partial}_A \psi = 0; \tag{13}$$

$$F_A^+ + \tau(\psi) = 0. \tag{14}$$

Its solutions are called *monopoles*.

Trace-free part

τ denotes the trace-free part.

$$\tau(\psi) = \psi\langle\psi, -\rangle - \|\psi\|^2 \,\mathrm{id} \tag{15}$$

with $\tau(\psi)(\psi) = 0$.

For a simplification, consider for any $n \times n$ matrix M the matrix:

$$\tau(M) := M - \frac{1}{n}\,\mathrm{tr}(M)\mathbf{1}_n \tag{16}$$

with $\mathrm{tr}\,\tau(M) = 0$.

Trace-free part

Another famous appearance of this expression is the *Einstein tensor* obtained by Ricci curvature:

$$G_{\mu\nu} := R_{\mu\nu} - \frac{R}{2} g_{\mu\nu}, \tag{17}$$

appearing in Einstein's field equations. But since 2 is the denominator, although 4 is the dimension of spacetime (hence all of the above are 4×4 matrices), it inverts the trace instead of getting rid of it. This corresponds to the fact, that a universe with negative energy density is positively curved and a universe with positive energy density is negatively curved.

Reducible monopoles

An anti-self dual Yang–Mills connection A (hence with $F_A^+ = 0$) and a vanishing spinor field $\psi = 0$ form a monopole. Such monopoles are called *reducible*, while solutions with non-vanishing spinor field are called *irreducible*.

Lichnerowicz formula

If (A, ψ) is a monopole, then the Lichnerowicz formula implies:

$$\not{\partial}_A^* \underbrace{\not{\partial}_A \psi}_{=0} = \nabla_A^* \nabla_A \psi + \frac{1}{4} R\psi + \frac{1}{2} \underbrace{F_A^+(\psi)}_{=\tau(\psi)=0}; \tag{18}$$

$$\|\nabla_A \psi\|^2 = \langle \psi, \nabla_A^* \nabla_A \psi \rangle = -\frac{1}{4} R \|\psi\|^2. \tag{19}$$

If the Ricci curvature of the base manifold is positive everywhere (meaning its energy density is negative everywhere when also solving Einstein's field equations), then the solution must be reducible with $\psi = 0$.

Gauge group

Let $\mathcal{A}$ denote the set of all pairs of vector potentials A and spinor fields ψ. It is more interesting to consider solutions up to describing the same physical situation (like for example functions up a constant describe the same derivative). One therefore considers the *gauge group* and *based gauge group* with any point $*$ given by:

$$\mathcal{G} = C^\infty(M, \mathrm{U}(1)) = \mathcal{G}_0 \times \mathrm{U}(1) \tag{20}$$

$$\mathcal{G}^* = \{g \in \mathcal{G} | g(*) = 1\} \tag{21}$$

$\mathcal{G}$ and therefore $\mathcal{G}_0$ act on $\mathcal{A}$ by rotation at every point. (More concretely, A is a $\mathrm{U}(1)$ connection and ψ a section of a complex vector bundle.) If $\psi = 0$, then it is therefore fully unaffected by this group action. Furthermore, the Seiberg–Witten equations are gauge invariant.

Seiberg–Witten moduli space

Let $\mathcal{B} = \mathcal{A}/\mathcal{G}$ and $\mathcal{B}^* = \mathcal{A}/\mathcal{G}^*$ be the orbit spaces, then there is a canonical projection $\mathcal{B}^* \twoheadrightarrow \mathcal{B}$. Let $\mathcal{M} \subset \mathcal{B}$ and $\mathcal{M}^* \subset \mathcal{B}^*$ be the *Seiberg–Witten moduli spaces* of monopoles, then there is a canonical projection $\mathcal{M}^* \twoheadrightarrow \mathcal{M}$.

Preimages of reducible monopoles are singletons while preimages of irreducible monopoles are U(1). Hence getting rid of all reducible solutions makes $\mathcal{M}^* \twoheadrightarrow \mathcal{M}$ into a principal U(1)-bundle.

Seiberg–Witten invariant

Unlike for the gauge groups, one doesn't necessarily have a direct product $\mathcal{M}^* = \mathcal{M} \times \mathrm{U}(1)$, but instead can have a global twist (similar to a Möbius strip, which also isn't $S^1 \times [0, 1]$), which is uniquely classified by the first Chern class.

This is the famous *Seiberg–Witten invariant*:

$$\mathrm{SW} = \left\langle c_1(\mathcal{M}^*)^{\frac{\dim \mathcal{M}}{2}}, [\mathcal{M}] \right\rangle \in \mathbb{Z}. \tag{22}$$

The Seiberg–Witten moduli space needing to have even dimensions translates into topological restrictions for the base manifold. It can be shown that the Seiberg–Witten invariant is independent of the Riemannian metric.

Seiberg–Witten conjecture

There are no known examples in which the Seiberg–Witten invariants don't vanish for a positive dimension of the Seiberg–Witten moduli space. Hence so far, computing the Seiberg–Witten invariants has always been counting points of zero-dimensional Seiberg–Witten moduli spaces with orientation.

Perturbed Seiberg–Witten equations

Avoiding reducible connections requires modifying the Seiberg–Witten equations. Let η be a self-dual form, which can not be the self-dual part of a curvature and whose existence also translates into topological restrictions for the base manifold, then the *perturbed Seiberg–Witten equations*:

$$\not{\partial}_A \psi = 0; \tag{23}$$

$$F_A^+ + \tau(\psi) = \eta \tag{24}$$

have no reducible solution as this would imply $\eta = F_A^+$, which is impossible by assumption.

Ricci curvature

As shown, positive Ricci curvature forces every monopole to be reducible and a suitable perturbation forces every solution to be irreducible. Therefore no solution can exist in such a scenario and since with the Seiberg–Witten moduli space is empty, all Seiberg–Witten invariants vanish.

If the base manifold has a non-vanishing Seiberg–Witten invariant, then it doesn't admit a Riemannian metric with overall positive Ricci curvature.

Smooth structures

Another important application of the Seiberg–Witten invariants is to distinguish smooth structures?, using the general idea, that these are detected by partial differential equations and then coded into their moduli space. Nonetheless, the smooth Poincaré conjecture in four-dimensions, asking if the four-dimensional sphere has exotic smooth structures, is still unknown. But an answer might come from physics again.