Dalian University of Technology (DUT)

Breakdown of a Breakthrough:

Navier-Stokes Existence and Smoothness Millennium Problem

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Navier–Stokes equations

Given a *density* $\rho \in C^\infty(\mathbb{R}^3 \times (0,\infty), \mathbb{R})$, *force field* $\mathbf{f} \in C^\infty(\mathbb{R}^3 \times (0,\infty), \mathbb{R}^3)$ and *viscosity* $\nu \in \mathbb{R}$, the *Navier–Stokes equations* are the conservation of momentum for a fluid with *velocity* $\mathbf{u} \in C^\infty(\mathbb{R}^3 \times [0,\infty), \mathbb{R}^3)$ with initial condition $\mathbf{u}_0 = \mathbf{u}(0,-)$ and *pressure* $p \in C^\infty(\mathbb{R}^3 \times [0,\infty), \mathbb{R})$ with initial condition $p_0 = p(0,-)$: [Eq. (1.1)]

$$\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}t} = \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} - \nu \Delta \mathbf{u} + \frac{\nabla p}{\rho} = \mathbf{f}. \tag{1}$$

$\mathrm{d}_t\mathbf{u}$ and $\partial_t\mathbf{u}$ are the change of the velocity over time when moving with the flow or at a fixed position, respectively. Their discrepancy $(\mathbf{u} \cdot \nabla)\mathbf{u}$ makes the Navier–Stokes equations non-linear.

$\Delta\mathbf{u}$ is the change in momentum due to friction, which makes the Navier–Stokes equation second order.

Viscosity

Any solution with $\nu = 1$ yields a solution with $\nu > 0$ by certain rescaling, which does not affect the singular time $t = 1$: [p. 7]

$$\widetilde{\mathbf{u}}(t, \mathbf{x}) = \sqrt{\nu}\mathbf{u}(\mathbf{x}/\sqrt{\nu}, t); \tag{2}$$

$$\widetilde{p}(t, \mathbf{x}) = \nu p(\mathbf{x}/\sqrt{\nu}, t); \tag{3}$$

$$\widetilde{\mathbf{f}} = \sqrt{\nu}\mathbf{f}(\mathbf{x}/\sqrt{\nu}, t). \tag{4}$$

Density

The *continuity equation* is the conservation of mass for the fluid. In integral form it states that for every volume V, one has:

$$\underbrace{\frac{\mathrm{d}}{\mathrm{d}t}\int_V \rho(t,\mathbf{x})\mathrm{d}\mathbf{x}}_{\text{change of mass in V}} = \oint_{\partial V} (\rho\mathbf{u})(t,\mathbf{x})\cdot \mathrm{d}\mathbf{s} \tag{5}$$

In differential form obtained with Gauss' law of integration, one has:

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} + \rho(\nabla\cdot\mathbf{u}) = \frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\mathbf{u}) = 0. \tag{6}$$

A fluid is called *incompressible* if $\rho = \mathrm{const}$, which implies $\nabla\cdot\mathbf{u} = 0$.

Navier–Stokes equations

In pure mathematics, the Navier–Stokes equations are therefore often expressed in a simpler form, which OpenAI calls the *residual*: [Eq. (3.1), p. 115]

$$\mathcal{R}(\mathbf{u}, p) := \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} - \Delta \mathbf{u} + \nabla p = \mathbf{f}. \tag{7}$$

OpenAI uses it to capture the non-linearity of the Navier–Stokes equation by perturbation: [p. 59, 105]

$$\begin{aligned} \mathcal{R}(\mathbf{u} + \mathbf{v}, p + q) = \mathcal{R}(\mathbf{u}, p) + \partial_t \mathbf{v} - \Delta \mathbf{v} + \nabla q \\ + (\mathbf{u} \cdot \nabla)\mathbf{v} + (\mathbf{v} \cdot \nabla)\mathbf{u} + (\mathbf{v} \cdot \nabla)\mathbf{v}. \end{aligned} \tag{8}$$

Non-linearity affects the velocity $\mathbf{u}$, however not the pressure p.

Navier–Stokes equations

Considering the divergence of the Navier–Stokes equations removes both terms $\partial_t u$ and Δu using the incompressibility condition $\nabla \cdot u = 0$. It yields: [p. 118]

$$\sum_{i,j=1}^{3} \partial_i \partial_j (u_i u_j) - \Delta p = \nabla \cdot \mathbf{f}. \tag{9}$$

Hence the pressure p satisfies the *Poisson equation* (inhomogenous Laplace equation). Two different solutions (u, p) and (u, q) for the same force field $\mathbf{f}$ yield the Laplace equation $\Delta(p - q) = 0$. Since p and q are assumed bounded for physical plausibility, Liouville's theorem implies $p - q = \mathrm{const}$ and therefore:

Any velocity $\mathbf{u}$ has a pressure p unique up to a constant if both solve the Navier–Stokes equation.[1]

[1] Although a solution to the Poisson equation for smooth inhomogenity always exists, for given $\mathbf{u}$ a solution p is not guaranteed to be bounded or for both to then solve the Navier–Stokes equations.

Claim

Original Claim: OpenAI gives a compactly supported solution $(\mathbf{u}, p)$ of the Navier–Stokes equations with a force field $\mathbf{f}$, vanishing initial conditions $\mathbf{u}_0 = \mathbf{u}(0, -) = \mathbf{0}$ and $p_0 = p(0, -) = 0$ as well as a blow-up at $t = 1$. This means:

- The solution $(\mathbf{u}, p)$ being compactly supported implies it automatically fulfills the first condition of being *physically plausible*. $\mathbf{u}_0$, $\mathbf{f}$ and all their derivatives vanish sufficiently fast when $t, \mathbf{x} \to \infty$. This prevents constructions in which energy is brought into the system from infinity and for eternity. It also enables the counterexample to descend from $\mathbb{R}^3$ to $T^3 = \mathbb{R}^3/\mathbb{Z}^3$.
- The residual $\mathcal{R}(\mathbf{u}, p) = \mathbf{f}$ must be smooth through the blow-up because the force field is. OpenAI reverses this to a **definition** $\mathbf{f} := \mathcal{R}(\mathbf{u}, p)$. [Eq. (10.5)] Therefore the Navier–Stokes equations are never actually solved, but the difficulty fully shifts to showing that the residual is smooth.

Claim

- The solution has bounded *kinetic energy* E, which the second part of being physically plausible, meaning *bounded* L^2 *norm*: [p. 1]

$$\sup_{t\in[0,1)} \|\mathbf{u}\|_{L^2(\mathbb{R}^3)} = \sup_{t\in[0,1)} \int_{\mathbb{R}^3} |\mathbf{u}(t,\mathbf{x})|^2 \mathrm{d}\mathbf{x} < E < \infty. \quad (10)$$

- The solution has a blow-up, meaning *diverging* L^∞ *norm*: [p. 1]

$$\lim_{t\to 1^-} \|\mathbf{u}\|_{L^\infty(\mathbb{R}^3)} = \lim_{t\to 1^-} \max\{|\mathbf{u}(t,\mathbf{x})| \,|\mathbf{x}\in\mathbb{R}^3\} = \infty. \quad (11)$$

Simplified Claim: OpenAI constructs compactly supported smooth functions $(\mathbf{u}, p)$ with vanishing initial conditions $\mathbf{u}_0 = \mathbf{u}(0,-) = \mathbf{0}$ and $p_0 = p(0,-) = 0$, so that the residual $\mathcal{R}(\mathbf{u}, p)$ is smooth, the kinetic energy finite and $\mathbf{u}$ has a blow-up at $t = 1$.

Proof

- **Prop. 9.9**: Construction of local functions $(\mathbf{u}_{\mathrm{loc}}, p_{\mathrm{loc}})$ causing the blow-up on a set Ω_* around the origin.
- **Prop. 10.1**: Expansion to functions $(\mathbf{u}, p)$ with support in a compact set $K \subset \mathbb{R}^3$, hence with $(\mathbf{u}, p)|_{\Omega_*} = (\mathbf{u}_{\mathrm{loc}}, p_{\mathrm{loc}})$.
- **Lem. 10.2**: Proof that for the force field $\mathbf{f} = \mathcal{R}(\mathbf{u}, p) \in C_c^\infty(\mathbb{R}^3 \times [0,1), \mathbb{R}^3)$ for every multiindex $\alpha \in \mathbb{N}^3$ and natural number $j \in \mathbb{N}$ the derivatives $\partial_x^\alpha \partial_t^j \mathbf{f}$ converge uniformly for $t \to 1^-$ and there exist functions $F_j \in C_c^\infty(\mathbb{R}^3, \mathbb{R})$ compactly supported in $K \subset \mathbb{R}^3$ with: [Eq. 10.6]

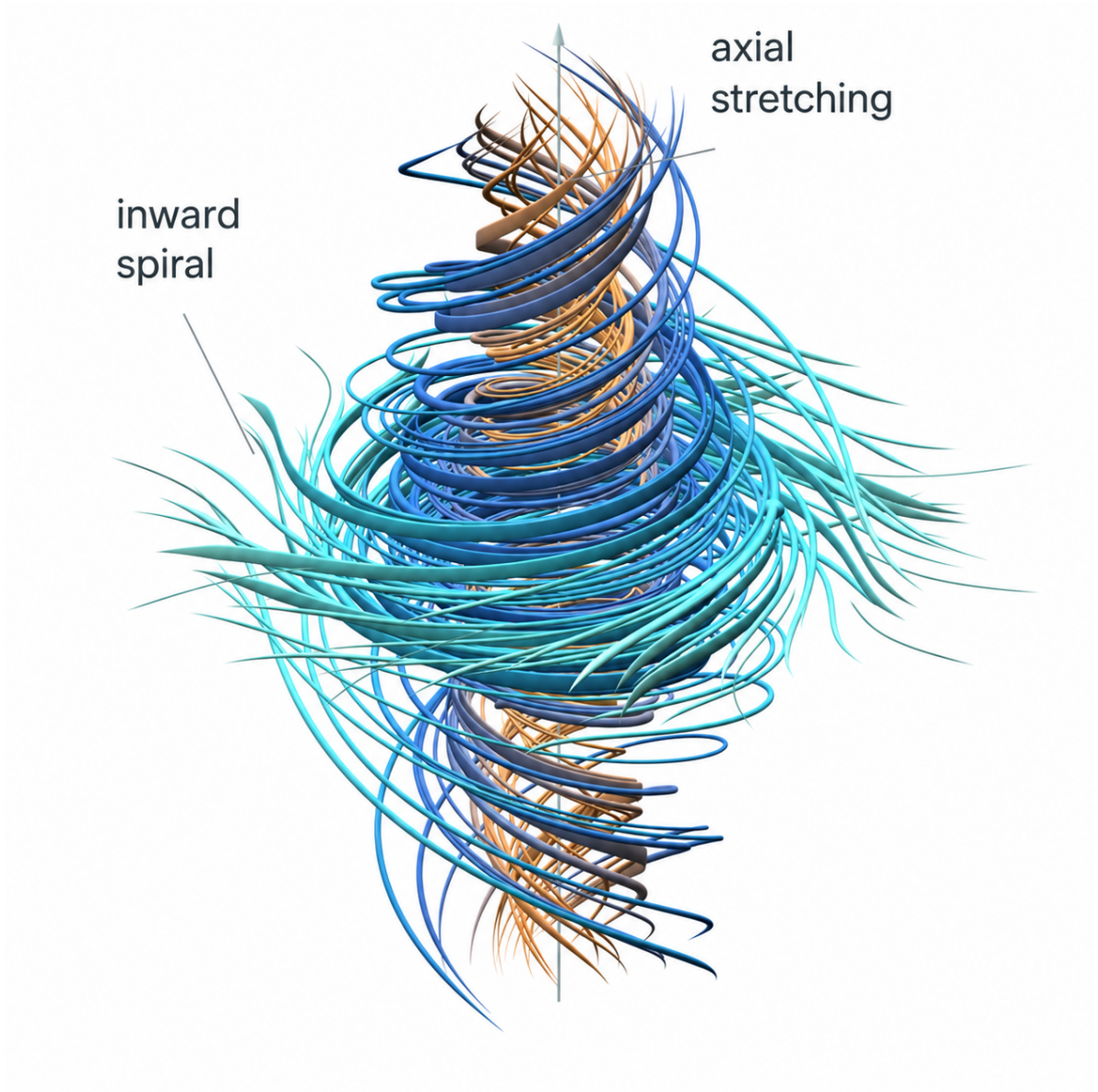
$$\lim_{t \to 1^-} \partial_x^\alpha \partial_t^j f(t,x) = \partial_x^\alpha F_j(x); \tag{12}$$

$$\partial_x^\alpha F_j(0) = 0. \tag{13}$$

Proof

- **Lem. 10.3**: Proof that the force field extends to $\mathbf{f} = \mathcal{R}(\mathbf{u}, p) \in C_c^\infty(\mathbb{R}^3 \times [0, \infty), \mathbb{R}^3)$ with support in $K \times [0, 2]$.
- **Lem. 10.4**: Proof that the kinetic energy is bounded.
- **Lem. 10.5**: Proof that despite the difficulty provided by non-linearity, the uniqueness of p for any given $\mathbf{u}$ also holds **turned around** as the uniqueness of $\mathbf{u}$ for the given p in the special case of the counterexample.

Construction



Construction

Let $\tau := 1 - t$ be the time until the singularity. Let ℓ_r be the *characteristic radius* and ℓ_z be the *characteristic height*, then there is a parameter $0 < h < 1/100$ with: [p. 8]

$$\ell_r \sim \tau^{1/2}; \tag{14}$$

$$\ell_z \sim \tau^{1/2-h}. \tag{15}$$

($\sim$ means that the expressions can be estimated against each other with just constants.) The spinning top gets thinner more rapidly than shorter with $\ell_r/\ell_z \sim \tau^h$. [p. 8]

Construction

Let $|u_r^{(0)}|$, $|u_\theta^{(0)}|$ and $|u_z^{(0)}|$ be the average velocities, then $|u_\theta^{(0)}|, |u_z^{(0)}| \sim \tau^{-1/2-h}$ and $|u_r^{(0)}| \in \mathcal{O}(\tau^{-1/2})$ with Landau notation. For the characteristic volume V, the characteristic velocity square $|u|^2$ and the total kinetic energy T, one has:

$$V \sim \ell_r^2 \ell_z \sim \tau^{3/2-h}; \tag{16}$$

$$|u|^2 = |u_r^{(0)}|^2 + |u_\theta^{(0)}|^2 + |u_z^{(0)}|^2 \sim \tau^{-1-2h}; \tag{17}$$

$$T \sim |u|^2 V \sim \tau^{1/2-3h}. \tag{18}$$

For $t \to 1^-$, hence $\tau \to 0^+$, the velocities diverge, although the kinetic energy T does not, with $T \to 0^+$.

Chapter 1: Navier–Stokes equations
Chapter 2: Navier–Stokes Existence and Smoothness Problem
Chapter 3: Chapter 1
Chapter 4: Chapter 10