Maxwell's equations

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Overview

- Vector analysis
- Maxwell's equations
- Wave equations

Vector analysis

Maxwell's equations are formulated using the formalism of vector analysis and in particular the *nabla operator*:

$$\nabla := \begin{pmatrix} \partial_x \\ \partial_y \\ \partial_z \end{pmatrix} \tag{1}$$

Gradient

The *gradient* transforms a scalar field $f \in C^1(\mathbb{R}^3, \mathbb{R})$ into a vector field:

$$\nabla f = \begin{pmatrix} \partial_x f \\ \partial_y f \\ \partial_z f \end{pmatrix} . \tag{2}$$

Rotation

The *rotation* transforms a vector field $\mathbf{A} \in C^1(\mathbb{R}^3, \mathbb{R}^3)$ into a vector field:

$$\nabla \times \mathbf{A} = \begin{pmatrix} \partial_y A_z \partial_z A_y \\ \partial_z A_x - \partial_x A_z \\ \partial_x A_y - \partial_y A_x \end{pmatrix}. \tag{3}$$

The rotation of a gradient always vanishes:

$$\nabla \times \nabla f = 0. \tag{4}$$

Divergence

The *divergence* transforms a vector field $\mathbf{A} \in C^1(\mathbb{R}^3, \mathbb{R}^3)$ into a scalar field:

$$\nabla \cdot \mathbf{A} = \partial_x A_x + \partial_y A_y + \partial_z A_z. \tag{5}$$

The divergence of a rotation always vanishes:

$$\nabla \cdot (\nabla \times \mathbf{A}) = 0. \tag{6}$$

Vector analysis

Let $\mathbf{A}$ be a vector field.

Gauß's theorem: For a volume V, one has:

$$\int_V \nabla \cdot \mathbf{A} \mathrm{d}^3 V = \oint_{\partial V} \mathbf{A} \cdot \mathrm{d}^2 \mathbf{S} \tag{7}$$

Stokes's theorem: For a surface A, one has:

$$\int_A (\nabla \times \mathbf{A}) \cdot \mathrm{d}^2 \mathbf{S} = \oint_{\partial A} \mathbf{A} \cdot \mathrm{d}\mathbf{s} \tag{8}$$

Electromagnetic field

The *electric field* and the *magnetic field* are described by vector fields:

$$\mathbf{E} = \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix}, \mathbf{B} = \begin{pmatrix} B_x \\ B_y \\ B_z \end{pmatrix} \tag{9}$$

The *electric charge density* is described by a scalar field ρ. The *electric current* is described by a vector field:

$$\mathbf{j} = \begin{pmatrix} j_x \\ j_y \\ j_z \end{pmatrix}. \tag{10}$$

First Maxwell's equation/Coulomb's law

Differential form:

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0}. \tag{11}$$

Integral form: Electric flux through the surface of a volume V corresponds exactly to the electric charge within. Hence it's not necessary to look within for that.

$$\frac{Q(V)}{\varepsilon_0} = \oint_{\partial V} \mathbf{E} \cdot \mathrm{d}^2\mathbf{S} \tag{12}$$

with the *electic charge*:

$$Q(V) = \int_V \rho \mathrm{d}^3 V. \tag{13}$$

Gauß's law gives the correspondence.

Second Maxwell's equation/Gauß's law

Differential form:

$$\nabla \cdot \mathbf{B} = 0. \tag{14}$$

Integral form: Magnetic flux through the surface of a volume V always vanishes. Hence magnetic flux lines are always closed.

$$\oint_{\partial V} \mathbf{B} \cdot \mathrm{d}^2\mathbf{S} = 0. \tag{15}$$

Gauß's law gives the correspondence.

Third Maxwell's equation/Faraday's law

Differential form:

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}. \tag{16}$$

Integral form: Changes in the magnetic field cause a electric vortex field. For a surface A, one has:

$$\oint_{\partial A} \mathbf{E} \cdot \mathrm{d}\mathbf{s} = -\frac{\partial}{\partial t} \int_A \mathbf{B} \cdot \mathrm{d}^2\mathbf{S}. \tag{17}$$

Stokes's law gives the correspondence.

Fourth Maxwell's equation/Ampére's law

Differential form:

$$\nabla \times \mathbf{B} = \mu_0 \left(\mathbf{j} + \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} \right). \tag{18}$$

Integral form: Electric and displacement currents cause a magnetic vortex field. For a surface A, one has:

$$\oint_{\partial A} \mathbf{B} \cdot \mathrm{d}\mathbf{s} = \mu_0 \left(I(A) - \varepsilon_0 \frac{\partial}{\partial t} \int_A \mathbf{E} \cdot \mathrm{d}^2\mathbf{S} \right) \tag{19}$$

with the *electric current*:

$$I(A) := \int_A \mathbf{j} \cdot \mathrm{d}^2\mathbf{S}. \tag{20}$$

Stokes's law gives the correspondence.

Maxwell's equations in a vacuum

In a vacuum without any electric charge ($\rho = 0$) and electric current ($\mathbf{j} = 0$), Maxwell's equations are:

$$\nabla \cdot \mathbf{E} = 0; \tag{21}$$

$$\nabla \cdot \mathbf{B} = 0; \tag{22}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}; \tag{23}$$

$$\nabla \times \mathbf{B} = -\mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}. \tag{24}$$

Since the divergence of a rotation always vanishes, applying the divergence on Faraday's and Ampére's law just leads to the derivative with repsect to time of Coulomb's and Gauß's law.

Maxwell's equations in a vacuum

Applying the rotaton on Faraday's and Ampére's law leads to wave equations:

$$\Delta \mathbf{E} = \nabla(\underbrace{\nabla \cdot \mathbf{E}}_{\overset{1.}{=}0}) - \nabla \times (\nabla \times \mathbf{E}) \overset{3.}{=} \nabla \times \frac{\partial \mathbf{B}}{\partial t} \overset{4.}{=} \mu_0 \varepsilon_0 \frac{\partial^2 \mathbf{E}}{\partial t^2}; \quad (25)$$

$$\Delta \mathbf{B} = \nabla(\underbrace{\nabla \cdot \mathbf{B}}_{\overset{2.}{=}0}) - \nabla \times (\nabla \times \mathbf{B}) \overset{4.}{=} -\mu_0 \varepsilon_0 \nabla \times \frac{\partial \mathbf{E}}{\partial t} \overset{3.}{=} \mu_0 \varepsilon_0 \frac{\partial^2 \mathbf{B}}{\partial t^2}. \quad (26)$$

Speed of light

The velocity c of electromagnetic waves is then given by:

$$\mu_0 \varepsilon_0 c^2 = 1. \tag{27}$$

" *This velocity is so nearly that of light, that it seems we have strong reason to conclude that light itself (including radiant heat, and other radiations if any) is an electromagnetic disturbance in the form of waves propagated through the electromagnetic field according to electromagnetic laws.*"
– James Clerk Maxwell

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As you already said, using all four Maxwell equations in a vacuum ($\rho = 0, \mathbf{j} = \mathbf{0}$), we get the wave equations:

$$\Delta \mathbf{E} = \nabla(\underbrace{\nabla \cdot \mathbf{E}}_{=0}) - \nabla \times (\nabla \times \mathbf{E}) = \nabla \times \frac{\partial \mathbf{B}}{\partial t} = \mu_0 \varepsilon_0 \frac{\partial^2 \mathbf{E}}{\partial t^2}$$

$$\Delta \mathbf{B} = \nabla(\underbrace{\nabla \cdot \mathbf{B}}_{=0}) - \nabla \times (\nabla \times \mathbf{B}) = -\mu_0 \varepsilon_0 \nabla \times \frac{\partial \mathbf{E}}{\partial t} = \mu_0 \varepsilon_0 \frac{\partial^2 \mathbf{B}}{\partial t^2},$$

which describe a wave propagating with the velocity $1/\sqrt{\mu_0 \varepsilon_0}$, whose value is exactly that of the speed of light, hence $\mu_0 \varepsilon_0 c^2 = 1$. James Maxwell commented this result with: "*This velocity is so nearly that of light, that it seems we have strong reason to conclude that light itself (including radiant heat, and other radiations if any) is an electromagnetic disturbance in the form of waves propagated through the electromagnetic field according to electromagnetic laws.*"

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2,296 ● 2 ● 11 ● 22

Maxwell's equations in a vacuum

Elementary solutions of the wave equations are given by:

$$\mathbf{E}(t, \mathbf{x}) = \mathbf{E}_0 \exp\left(i(\omega t + \mathbf{k} \cdot \mathbf{x})\right); \tag{28}$$

$$\mathbf{B}(t, \mathbf{x}) = \mathbf{B}_0 \exp\left(i(\omega t + \mathbf{k} \cdot \mathbf{x})\right). \tag{29}$$

Faraday's and Ampére's law, which compare the elementary waves, assures their *wave frequency* ω and *wave vectors* $\mathbf{k}$ are identical. Both wave equations lead to the conditions:

$$\omega = c\|\mathbf{k}\|. \tag{30}$$

Maxwell's equations in a vacuum

Maxwell's equations then yield the conditions:

$$\mathbf{k} \cdot \mathbf{E}_0 = 0; \tag{31}$$

$$\mathbf{k} \cdot \mathbf{B}_0 = 0; \tag{32}$$

$$\mathbf{k} \times \mathbf{E}_0 = \omega \mathbf{B}_0; \tag{33}$$

$$\mathbf{k} \times \mathbf{B}_0 = \mu_0 \varepsilon_0 \omega \mathbf{E}_0. \tag{34}$$

All together show that the three vectors $\mathbf{E}_0$, $\mathbf{B}_0$ and $\mathbf{k}$ are all orthogonal to each other with:

$$\|\mathbf{E}_0\| = c\|\mathbf{B}_0\|. \tag{35}$$

Maxwell's equations in a vacuum

A general solution of Maxwell's equations is then given by:

$$\mathbf{E}(t, \mathbf{x}) = \int_{\mathbb{R}^3} \mathbf{E}_0(\mathbf{k}) \exp\left(i(c\|\mathbf{k}\|t + \mathbf{k} \cdot \mathbf{x})\right) \mathrm{d}^3\mathbf{k}; \tag{36}$$

$$\mathbf{B}(t, \mathbf{x}) = \int_{\mathbb{R}^3} \mathbf{B}_0(\mathbf{k}) \exp\left(i(c\|\mathbf{k}\|t + \mathbf{k} \cdot \mathbf{x})\right) \mathrm{d}^3\mathbf{k}. \tag{37}$$

with $\mathbf{E}_0(\mathbf{k})$, $\mathbf{B}_0(\mathbf{k})$ and $\mathbf{k}$ all orthogonal and $\|\mathbf{E}_0(\mathbf{k})\| = c\|\mathbf{B}_0(\mathbf{k})\|$.