Einstein's field equations

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Contradiction

Newton's theory of gravity (1687):

Mass causes gravity. Motion and rotation don't.

Einstein's theory of special relativity (1905):

Energy and mass are equivalent:

$$E = mc^2. \tag{1}$$

Gravitoelectromagnetism

A simple first try would be to transfer electrodynamics, which is consistent with special relativity, meaning it considers movement in a Lorentz-invariant way:

Resting electric charge causes an electric field.
Moving electric charge, an electric current, causes a magnetic field.

From this perspective, it looks like Newton's theory gravity can be saved by just adding another field:

Resting mass causes a "gravitoelectric" field,
which is simply the usual gravitational field.
Moving mass causes a "gravitomagnetic" field.

Unfortnutely, it wasn't that easy and took an entire decade to be fixed. Nonetheless, general relativity in first order still fully resembles electromagnetism, meaning that the naming is still used.

Gravitational waves

In electrodynamics, Maxwell's equations in a vacuum combine to wave equations for the electromagentic field.

In the analogue theory for gravity, Maxwell's equations also produce gravitoelectromagnetic (gravitational) waves.

Gravitational waves

Electromagentic waves don't carry any electric charge or any electric current.

Gravitational waves, on the other hand, like every wave carry energy, which is equivalent to a mass and hence is a source of gravity again. Again, Einstein's formula leads back to the initial problem of there being a previously unconsidered source of gravity.

Since gravity now becomes its own source, any attempt of a working theory must describe gravity by a non-linear equation.

Minkowski tensor

Flat spacetime is described by the Minkowski tensor, a 4×4 matrix $\eta = \text{diag}(+1, -1, -1, -1)$, which means that the spacetime length s of a spacetime 4-vector x fulfills:

$$s^2 = x^T \eta x. \tag{2}$$

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Due to the negative signs in the Minkowski matrix, it is possible for s^2 to be negative. Such a case corresponds to the spatial distance being larger than the temporal distance and hence a velocity larger than the speed of light.

Einstein notation

Like in special relativity, index notation is usually used and summing over indices that appear twice. For example the formula for the spacetime length becomes:

$$s^2 = \eta_{\mu\nu} x^\mu x^\nu. \tag{3}$$

Inverse matrices are denoted with indices swapped from the bottom (covariant) to the top (contravariant).

Metric tensor

General relativity now replaces the Minkowski metric, a 4×4 matrix η, by the more general metric tensor, a 4×4 matrix g, which is also invertible and symmetric. In Newton's theory of gravity, in which the gravitational potential Φ is the only data, it has the following form:

$$g = \text{diag}\left(1 + \frac{2\Phi}{c^2}, -1, -1, -1\right). \tag{4}$$

An important result here is that gravity not only vanishes for $\Phi \to 0$, but also for $c \to \infty$.

Geodesic equation

Obtaining this description of Newton's theory of gravity doesn't even require its field equation $\Delta\Phi = 4\pi G\rho$, but instead only the equations $\ddot{\mathbf{x}} = -\nabla\Phi(\mathbf{x})$. In Riemannian geometry, these become the *geodesic equations*:

$$\ddot{x}^\sigma = \Gamma^\sigma_{\mu\nu}\dot{x}^\mu\dot{x}^\nu. \tag{5}$$

They are exactly the Euler–Lagrange equations obtained by the action over the previous formula of the spacetime length:

$$S = \int \mathrm{d}s = \int \sqrt{g_{\mu\nu}(x)\mathrm{d}x^\mu \mathrm{d}x^\nu} = \int \sqrt{g_{\mu\nu}(x)\dot{x}^\mu\dot{x}^\nu}\,\mathrm{d}t. \tag{6}$$

Christoffel symbol

Hence the geodesic equations minimize the length of its solutions and are connected to the metric tensor by the Christoffel symbol:

$$\Gamma^{\sigma}_{\mu\nu} = \frac{1}{2} g^{\sigma\kappa} \left(\partial_{\mu} g_{\kappa\nu} + \partial_{\nu} g_{\kappa\mu} - \partial_{\kappa} g_{\mu\nu} \right). \tag{7}$$

Curvature of spacetime

Although the Christoffel symbols represent the curvature of spacetime in the geodesic equations, it is not invariant under coordiante changes and therefore not a Riemann tensor. Nonetheless, suitable combinations of it are:

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a can be determined by considering the Newtonian limit and b can be considered a free parameter describing dark energy, then more usually denoted as the *cosmological constant* Λ.

Einstein's field equations

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$$T = \operatorname{diag}(\rho, 0, 0, 0)\,. \tag{13}$$

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with Einstein's gravitational constant:

$$\kappa = \frac{8\pi G}{c^4}. \tag{15}$$

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Let $T = g^{\mu\nu} T_{\mu\nu}$ be the trace of the energy–stress tensor, then Einstein's field equations can be brought into a dual form. Applying $g^{\mu\nu}$ to them yields:

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Solving for R and putting it into Einstein's field equations yields:

$$R_{\mu\nu} - \Lambda g_{\mu\nu} = \kappa \left(T_{\mu\nu} - \frac{T}{2} g_{\mu\nu} \right). \tag{17}$$