From the Schrödinger and Klein–Gordon equation to the Dirac equation

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Basic formulas

In 1900, Max Planck introduced a "helping" variable to consider the arising necessecity to quantize the thermal energy of black body radiation. It became known as *reduced Planck's constant* $\hbar$.

In 1905, Albert Einstein expanded this argument in his Nobel Prize winning work on the photoelectric effect to also hold for electromagnetic energy E depending on the frequency ω. It became known as the *Planck–Einstein formulas*:

$$E = \hbar\omega. \tag{1}$$

In 1924, Louis de Broglie assumed in his PhD thesis that there is no fixed distinction between matter described by a momentum $\mathbf{p}$ and waves described by a wave vector $\mathbf{k}$, so that both can even be related. It became known as the *de Broglie formula*:

$$\mathbf{p} = \hbar\mathbf{k}. \tag{2}$$

An elementary wave with frequency ω and wave vector $\mathbf{k}$ can then be expressed through its energy E and momentum $\mathbf{p}$ using:

$$\psi(t,x) = \exp\left(i(\omega t + \mathbf{k}\cdot\mathbf{x})\right) = \exp\left(\frac{i}{\hbar}(Et + \mathbf{p}\cdot\mathbf{x})\right). \tag{3}$$

In quantum mechanics, energy E and momentum $\mathbf{p}$ are then repleced by differential operators:

$$E\psi(t,\mathbf{x}) = -i\hbar\frac{\partial}{\partial t}\psi(t,x); \tag{4}$$

$$\mathbf{p}\psi(t,\mathbf{x}) = -i\hbar\nabla\psi(t,\mathbf{x}). \tag{5}$$

Schrödinger equation

The non-relativistic energy-momentum relation is given by:

$$E = \frac{p^2}{2m} + V. \tag{6}$$

In 1925, Erwin Schrödinger replaced $E \rightsquigarrow -i\hbar\partial_t$ and $\mathbf{p} \rightsquigarrow -i\hbar\nabla$ and applied it to a wave function ψ, which yields the *Schrödinger equation*:

$$-i\hbar\frac{\partial}{\partial t}\psi = -\frac{\hbar^2}{2m}\Delta\psi + V\psi. \tag{7}$$

Mathematically, the Schrödinger equation is the heat equation with an additional potential term. It cannot be used for relativistic particles since it isn't Lorentz invariant. (It contains a first order derivative with respect to time and second order derivatives with respect to space, which conflicts with them being mixed.)

Klein–Gordon equation

The relativistic energy-momentum relation is given by:[1]

$$E^2 = (pc)^2 + (m_0 c^2)^2 \tag{8}$$

In 1926, Oscar Klein and Walter Gordon replaced $E \rightsquigarrow -i\hbar\partial_t$ and $\mathbf{p} \rightsquigarrow -i\hbar\nabla$ and applied it to a wave function ϕ, which yields the *Klein–Gordon equation*:

$$\left(\Box + \left(\frac{m_0 c}{\hbar}\right)^2\right)\phi = 0. \tag{9}$$

with the *wave operator*:

$$\Box := \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \Delta = \eta^{\mu\nu}\partial_\mu\partial_\nu. \tag{10}$$

Mathematically, the Klein–Gordon equation is the wave equation with an additional mass term.

[1] In fact, this is simply Einstein' formula $E = mc^2$ rewritten using the relativistic mass $m = \gamma m_0$.

BPS equations

Bogomolny–Prasad–Sommerfield (BPS) equations are PDE of first order, which imply more complicated PDE of second order.

Due to the second derivative with respect to time and a failure to fit observations, relativistic alternatives to the Klein–Gordon equation needed to be found.

In 1927, Paul Dirac attempted to find a BPS equation of the Klein–Gordon equation.

Dirac operators

A Dirac operator ∂ is a partial differential operator of first order, which when applied twice yields a (generalized) Laplace operator:

$$\Delta = \partial^2. \tag{11}$$

Examples of Dirac operators

A simple Dirac operator $C^2(\mathbb{C}) \to C^2(\mathbb{C})$ is given by:

$$\begin{pmatrix} 0 & \partial_x - i\partial_y \\ \partial_x + i\partial_y & 0 \end{pmatrix}^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \Delta. \tag{12}$$

For a smooth manifold M with exterior derivative d and its adjoint δ, a simple Dirac operator is $\mathrm{d} + \delta \colon \Omega(M) \to \Omega(M)$ (combining differential forms of all degrees) since:

$$\Delta = \mathrm{d}\delta + \delta\mathrm{d} \tag{13}$$

and $\mathrm{d}^2 = 0$ as well as $\delta^2 = 0$ (which is used for de Rham cohomology).

For any operator D with adjoint D^*, so that $\Delta = DD^* = D^*D$, a Dirac operator can be created by:

$$\begin{pmatrix} 0 & D \\ D^* & 0 \end{pmatrix}^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \Delta. \tag{14}$$

Dirac operators

Finding Dirac operators is a differential problem, but it can be turned into an algebraic problem. As is well-known, the Fourier transformation transfers partial derivatives into simple multiplications:

$$\psi(t,x) = \int_{\mathbb{R}} \int_{\mathbb{R}^3} \widetilde{\psi}(\omega, \mathbf{k}) \exp\left(\omega t + \mathbf{k} \cdot \mathbf{x}\right) \mathrm{d}\omega \mathrm{d}^3\mathbf{k} \qquad (15)$$

Symbols of Dirac operators

Partial differential operators can be studied using their *symbols*, which replace derivatives by vectors. (More generally, symbols map to endomorphisms of vector bundles.) In case of Dirac operators, a strongly simplified outline gives for its symbol:

$$\sigma_\partial(v)^2 = \sigma_\Delta(v) = |v|^2 \tag{16}$$

$$\begin{aligned}\{\sigma(v), \sigma(w)\} &= \sigma(v)\sigma(w) + \sigma(w)\sigma(v) \\ &= \underbrace{(\sigma(v) + \sigma(w))^2}_{=\sigma(v+w)^2} - \sigma(w)^2 - \sigma(v)^2 \\ &= |v + w|^2 - |v|^2 - |w|^2 = 2\langle v, w\rangle \end{aligned} \tag{17}$$

for all vectors v and w. (σ_∂ in the latter equation replaced by σ for simplicity.) Such a relation is called a *Clifford structure*, which exists most generally for vector spaces with a scalar product.

Dirac matrices

Such a Clifford structure also appears for the Dirac operator, which was constructed for the Dirac equation:

$$\partial = \gamma^\mu \partial_\mu; \tag{18}$$

$$\eta^{\mu\nu}\partial_\mu\partial_\nu = \Box = \partial^2 = \gamma^\mu\gamma^\nu\partial_\mu\partial_\nu = \frac{1}{2}\{\gamma^\mu, \gamma^\nu\}\partial_\mu\partial_\nu$$
$$\Rightarrow \{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}. \tag{19}$$

γ^0, γ^1, γ^2 and γ^3 are the 4×4 *Dirac matrices* (or *Gamma matrices*). Using the above Dirac operator requires expanding the wave function to a wave vector, which is called *Dirac spinor*. Now the Dirac equation becomes:

$$(i\hbar\partial - m)\psi = 0. \tag{20}$$

Applying $i\hbar\partial + m$ yields the Klein–Gordon equation for every component of the Dirac spinor.

Spin structures

M is *orientable* if and only if $w_1(M) = 0$ if and only if the restriction of the tangent bundle on every embedded circle $S^1 \hookrightarrow M$ is trivial.

M is *spin* or *spinnable* if and only if $w_1(M) = 0$ and $w_2(M) = 0$ if and only if the restriction of the tangent bundle on every embedded circle $S^1 \hookrightarrow M$ and every embedded sphere $S^2 \hookrightarrow M$ is trivial.